Direct measurement of finite-time disentanglement induced by a reservoir

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We propose a method for directly probing the dynamics of disentanglement of an initial two-qubit entangled state, under the action of a reservoir. We show that it is possible to detect disentanglement, for experimentally realizable examples of decaying systems, through the measurement of a single observable, which is invariant throughout the decay. The systems under consideration may lead to either finite-time or asymptotic disentanglement. A general prescription for measuring this observable, which yields an operational meaning to entanglement measures, is proposed, and exemplified for cavity quantum electrodynamics and trapped ions.

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Entanglement is the most characteristic trait of quantum mechanics [1]. As such, it has led not only to intense debate since the beginnings of quantum mechanics [2], but also to a variety of possible applications, ranging from communications [3, 4] to computation [5]. Decoherence and entanglement are closely connected phenomena: not only decoherence follows from the entanglement of the system of interest with the rest of the Universe [6], but also it is responsible for the fragility of entanglement in systems interacting with reservoirs. Because of this, understanding the basic mechanisms of decoherence and disentanglement has both fundamental and practical implications. However, even for simple systems of two qubits, the available entanglement measures, like concurrence as introduced by Wootters[7], involve mathematical operations that do not seem to have a direct physical interpretation. It is desirable, therefore, to find examples of systems for which an entanglement measure could be associated to an observable that could be easily measured. This would not only make it easier to follow the dynamics of disentanglement for systems interacting with reservoirs, but it could be helpful for the physical interpretation of this process. Here we show that, for an experimentally realizable example of a decaying system, it is possible to directly measure concurrence, through the detection of an observable that is invariant throughout the evolution of the system.

A state of a bipartite system is disentangled or separable if it can be written in the form

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B , \quad (p_i \geq 0) . \tag{1}$$

For a pair of qubits, described by the density operator ρ, entanglement may be quantified by the concurrence[7]

$$\mathcal{C}(\hat{\rho}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} , \tag{2}$$

where the λ_i's are the eigenvalues, in decreasing order, of the Hermitian matrix $\hat{\rho}(\hat{\sigma}_y \otimes \hat{\sigma}_y)\hat{\rho}^*(\hat{\sigma}_y \otimes \hat{\sigma}_y)$. It ranges from $\mathcal{C} = 0$ for a separable state to $\mathcal{C} = 1$ for a maximally entangled state.

The connection between disentanglement of bipartite systems and decoherence has been recently analyzed, within the framework of models that may lead to finite-time disentanglement, while the individual subsystems decay asymptotically in time[8, 9, 10, 11, 12]. These examples are interesting insofar as they allow a clear separation between the disentangling and the decay dynamics. In [10], a realistic system was considered, consisting of a pair of two-level atoms, each interacting with its own reservoir. Depending on the initial atomic state, one may have either finite-time or asymptotic disentanglement.

Simple methods to detect this transition from entanglement to disentanglement would be clearly desirable. However, the non-physical nature of the operations involved in the definition of concurrence imply that such a detection is usually a challenging problem, requiring the full tomographic reconstruction of the state. Entangled states may also be identified through entanglement witnesses, which are non-positive operators that are positive in the subspace of separable states[13]. Entanglement witnesses allow one to identify some, but not all, entangled states.

For two trapped ions, entanglement between corresponding two-level internal states has been established either by means of a witness involving a single matrix element of the two-qubit density operator [14, 15], or by measuring the full concurrence via tomographic reconstruction of the state [16].

In this paper, we show that it is actually possible to detect, through the measurement of a single observable, finite-time disentanglement of an initial pure state of two qubits, which evolves under the action of a reservoir. This observable is a "perfect witness" for the class of states here considered: any entangled state in this class would lead to a negative value for this operator, which is precisely equal to minus the concurrence

of the state. Remarkably, the same observable yields the concurrence throughout the evolution of the system. It is possible therefore, by measuring this operator as the system evolves, to pinpoint the precise moment when disentanglement occurs. Also, a simple physical interpretation can be given to concurrence in this case. Our proposal is within present experimental capabilities in systems like trapped ions[14, 15], cavity quantum electrodynamics (cavity QED)[17, 18], circuit quantum electrodynamics[19], and nuclear magnetic resonance[20].

We consider here initial states of the form $|\Psi(0)\rangle = |\alpha||00\rangle + |\beta|\exp(i\theta)|11\rangle$, where 0 and 1 correspond to the ground and excited state of each qubit, respectively. The two qubits may stand for spin-down and up internal states of two trapped ions, or to one- and zero-photon states of two modes of the electromagnetic field in the same or in different cavities. States like this have been realized in experiments with trapped ions [15, 16, 21, 22]. The phase θ is attributed by the preparation process. We assume that the two qubits evolve under the influence of low-temperature independent and identical reservoirs, so that higher-energy states are not populated.

Under these conditions, we show that the concurrence of the density operator evolving from the above initial state is equal to $\mathcal{C}(t) = \max\{0, -W(t)\}$, where $W(t) = \mathrm{Tr}\left[\hat\rho(t)\hat W_\theta\right]$, and $\hat W_\theta = 1 - 2|\Phi(\theta)\rangle\langle\Phi(\theta)|$, with

$$|\Phi(\theta)\rangle = (|00\rangle + e^{i\theta}|11\rangle)/\sqrt{2}\,. \tag{3}$$

The observable $\hat W_\theta$ is a "perfect witness," the same throughout the evolution of the system: it is positive for separable states, and negative for entangled states.

We note that $W(t) = 1-2P(t)$, where $P(t)$ is the probability of finding the system in the state $|\Phi(\theta)\rangle$ at time t. It is easy to show that other choices of the relative phase in (3), different from θ, would yield smaller values of the corresponding probability. This yields a simple physical interpretation of concurrence in this case: it is twice the maximal excess probability, with respect to 50%, of finding the state of the system in a maximally entangled state corresponding to the subspace spanned by $\{|00\rangle, |11\rangle\}$. One should note that evaluation of $W(t)$ amounts to measuring this probability. We show now that this can be done in a simple way, by inverting the process which yields $|\Phi(\theta)\rangle$ from the state $|00\rangle$.

State (3) may be obtained from the state $|00\rangle$ by applying to it an unitary transformation $\hat U(\theta)$ composed of a one-qubit $\pi/2$ rotation followed by a CNOT operation. Therefore, the probability $P(t)$ can be written as

$$P(t) = \langle\Phi(\theta)|\hat\rho(t)|\Phi(\theta)\rangle = \langle 00|\hat U^{-1}(\theta)\hat\rho\hat U(\theta)|00\rangle\,, \tag{4}$$

that is, the probability of finding the system in the state (3) can be obtained by applying to the system the inverse of $\hat U(\theta)$, and then detecting the probability of finding both qubits in the ground state.

The general format, in the basis $|11\rangle, |10\rangle, |01\rangle, |00\rangle$, of the time-dependent density matrix that evolves from state $|\Psi(0)\rangle$ is:

$$\hat\rho(t) = \begin{pmatrix} w(t) & 0 & 0 & z(t) \\ 0 & x(t) & 0 & 0 \\ 0 & 0 & x(t) & 0 \\ z^*(t) & 0 & 0 & y(t) \end{pmatrix} \tag{5}$$

with $w(t)$, $x(t)$, and $y(t)$ real. Indeed, the incoherent decay of the state $|11\rangle$ under the action of the reservoir does not lead to coherence between the states $|10\rangle$ and $|01\rangle$. Also, the symmetry of the initial state and the equality of the damping rates for both reservoirs implies that the population of theses two states is always the same.

The concurrence corresponding to this state is easily determined as $C(t) = \max\{0, 2|z(t)| - 2x(t)\}$. Therefore, the state is entangled if and only if $|z(t)| > x(t)$.

The probability of finding the system described by (5) in the state (3) is given by

$$P(t) = [1 - 2x(t) + 2|z(t)|]/2\,, \tag{6}$$

and therefore $\mathcal{C}(t) = \max\{0, 2P(t) - 1\}$, as anticipated.

For the initial state $|\Psi(0)\rangle$, $\mathcal{C}(0) = 2|\alpha\beta^*| = 2|z(0)|$, which equals one for the maximally entangled state $(|00\rangle + e^{i\theta}|11\rangle)/\sqrt{2}$. As time evolves, the state becomes disentangled when $P(t)$ reaches the value $1/2$.

This result is rather insensitive to small deviations from the above state. Thus, if the populations of states $|10\rangle$ and $|01\rangle$ are slightly different (due, for instance, to unequal decay rates), then if $\epsilon(t) = |x_1(t) - x_2(t)| \ll |z(t)|$ it is easy to show that the concurrence is still given by the above expression, up to terms of order ϵ^2. Also, our method can be extended to any state in the subspace $\{|00\rangle, |11\rangle\}$.

We show now that, for the usual linear coupling with markoffian reservoirs, $\mathcal{C}(t)$ may reach zero at finite times. The master equation governing the time behavior of the system can be written in the Lindblad form [23]:

$$\dot{\hat\rho} = \sum_i (\gamma_i/2)\left(2\hat c_i\hat\rho\hat c_i^\dagger - \hat c_i^\dagger\hat c_i\hat\rho - \hat\rho\hat c_i^\dagger\hat c_i\right), \tag{7}$$

where γ_i are decay rates and $\hat c_i, \hat c_i^\dagger$ are operators representing the coupling to the reservoir. For atoms decaying spontaneously due to the coupling to a zero temperature reservoir, $\hat c = \hat\sigma_-$, where $\hat\sigma_-$ is the lowering operator acting on the electronic levels of the atom. For modes of the electromagnetic field in one or two cavities, coupled to a zero- temperature reservoir, the decaying dynamics is analogous to the ionic system, provided that the photonic state is initially in the subspace $\{|0\rangle, |1\rangle\}$. In this case, $\hat c_i$ correspond to annihilation operators for photons, for each of the two cavity modes. The solution of (7) for

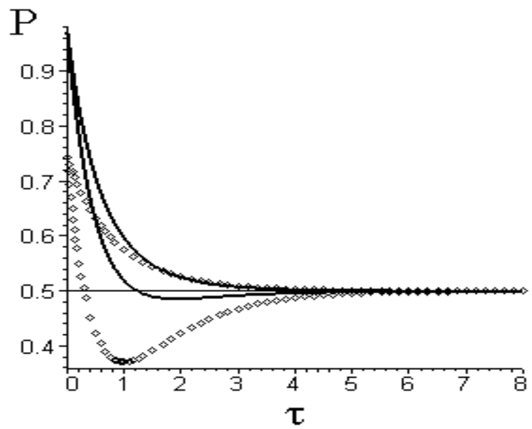

FIG. 1: P as a function of $\tau = \gamma t$ for four different initial states, of the form $\alpha|00\rangle + \beta|11\rangle$, with concurrence $\mathcal{C}(0) = 2|\alpha||\beta|$. Solid thick lines correspond to $\mathcal{C}(0) = \sqrt{2}/3$, while dotted lines correspond to $\mathcal{C}(0) = \sqrt{15}/16$. Concurrence, as a function of time, is given by $\max\{0, 2P(t) - 1\}$. Entanglement persists as long as $P > 1/2$ (this bound corresponds to the thin solid line). States with the same initial entanglement exhibit finite-time disentanglement for $|\beta| > |\alpha|$ and infinite-time disentanglement for $|\alpha| > |\beta|$

any initial state lying in the subspace $\{|00\rangle, |11\rangle\}$ is of the form (5) with:

$$\begin{aligned} w(t) &= w(0)e^{-2\gamma t}, \\ x(t) &= w(0)(1 - e^{-\gamma t})e^{-\gamma t}, \\ y(t) &= y(0) + w(0)(1 - e^{-\gamma t})^2, \\ z(t) &= z(0)e^{-\gamma t}. \end{aligned} \tag{8}$$

Consequently, an initial state of the form $\alpha\,|00\rangle + \beta\,|11\rangle$ becomes separable for $t_s = -\frac{1}{\gamma}\ln(1 - \frac{|\alpha|}{|\beta|})$. It is clear that only for states with $|\beta| > |\alpha|$ is this condition fulfilled for finite times. The probability $P(t)$ is shown in Fig. 1, for four different initial states. One should note that states with the same initial entanglement exhibit finite-time disentanglement for $|\beta| > |\alpha|$ and infinite-time disentanglement for $|\alpha| > |\beta|$. For states of the form $\gamma|01\rangle + \delta|10\rangle$, disentanglement occurs asymptotically in time. This justifies our interest in the special class of states here considered, since they allow a clear separation between the processes of disentanglement and decoherence, besides allowing a simple monitoring procedure of the disentanglement process.

One should note that the method of measurement outlined above is valid only if the operations on the qubits do not involve auxiliary systems, as is the case for instance in nuclear magnetic resonance. On the other hand, the realization of quantum gates in ion traps (where the vibrational mode is used for the realization of a CNOT gate), or in cavity QED (where the cavity mode mediates the interaction between two atoms, or an atom mediates the interaction between two cavity modes) require auxiliary systems. In these cases, the operation $\hat{U}(\theta)$ does not depend only on the qubit operators, so that Eq. (4) does not hold. In spite of this, the above method can still be used to measure entanglement, the results differing from the concurrence by a scaling constant. We exemplify this with application to ion traps and cavity QED.

For two trapped ions, the production of the initial state and the measurement of $P(t)$ can be done by employing standard techniques involved in the production and detection of entangled states [15, 21, 22].

In these experiments, maximally entangled states are produced through a combination of red-sideband and blue-sideband transitions and detected using the shelving technique: the ion undergoes cycling transitions to an excited unstable state if and only if it is in the ground state, the resulting fluorescence being measured with very high detection efficiency (of the order of 99%). The fluorescence yield of each ion is thus proportional to the probability of finding it in the ground state.

One should note that the decay of the ions does not lead to population of the vibrational modes, which remain in the ground state. We consider the initial state $(|\alpha||g_1g_20\rangle + e^{i\theta}|\beta||e_1e_20\rangle)/\sqrt{2}$, where e_i and g_i stand respectively for the upper and lower internal states of the ions ($i = 1, 2$) and integer numbers stand for the states of the vibrational mode. Disentanglement of this state can be measured by applying to the first ion a red-sideband pulse, so that $|g_11\rangle \to |e_10\rangle$ and $|e_10\rangle \to -|g_11\rangle$. This operation transforms the state $\hat{\rho}(t) \otimes |0\rangle\langle 0|$, with $\hat{\rho}(t)$ given by (5), into

$$\begin{aligned} & w(t)\,|g_1e_21\rangle\langle g_1e_21| + y(t)\,|g_1g_20\rangle\langle g_1g_20| \\ & -z(t)\,|g_1e_21\rangle\langle g_1g_20| - z^*(t)\,|g_1g_20\rangle\langle g_1e_21| \\ & +x(t)(|g_1e_20\rangle\langle g_1e_20| + |g_1g_21\rangle\langle g_1g_21|)\,. \end{aligned}$$

Next a blue-sideband pulse is applied on the second ion, so that $|g_20\rangle \to (|g_20\rangle - \exp(i\delta)\,|e_21\rangle)/\sqrt{2}$, $|e_21\rangle \to (\exp(i\delta)\,|g_20\rangle + |e_21\rangle)/\sqrt{2}$, and $|g_21\rangle \to \cos(\pi\sqrt{2}/4)\,|g_21\rangle - \sin(\pi\sqrt{2}/4)\exp(-i\delta)\,|e_22\rangle$. The probability $P_{gg}(\delta, t)$ that both ions are in the ground state is

$$P_{gg}(\delta, t) = 1/2 - |z(t)|\cos(\theta - \delta) - \eta x(t)\,, \tag{9}$$

where $\eta = \sin^2(\pi\sqrt{2}/4)$. Choosing δ so that $\cos(\theta - \delta) = -\eta$ one gets, calling $\mathcal{P}_{gg}(t)$ the value of $P_{gg}(\delta, t)$ at this point:

$$2\mathcal{P}_{gg}(t) - 1 = \eta\,[2P(t) - 1]\,, \tag{10}$$

with $P(t)$ given by (6). This shows that $\max\{0, 2\mathcal{P}_{gg} - 1\}$ is proportional to the concurrence. One should note that even if the initial phase θ is not known, measurement of $P_{gg}(\delta, t)$ for three values of the phase δ, leading to three linearly independent equations for $x(t)$, $|z(t)|$, and θ, would allow one to determine the concurrence.

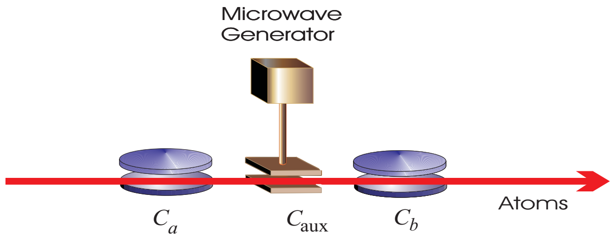

FIG. 2: Experimental setup for probing finite-time decoherence in cavity QED.

For cavity QED, the experimental setup involves two high-Q cavities (C_a and C_b) on either side of a low-Q cavity (C_{aux}), as shown in Fig. 2. A two-level atom crosses this system, interacting resonantly with the three cavities. The field in C_{aux} can be taken as classical, generating a π rotation of the atomic state. The fields in C_a and C_b are initially in the vacuum state. The interaction time between the atom and each cavity is adjusted, for open-cavity geometry[17, 18], by Stark-shifting the relevant atomic levels, so as to tune them into resonance with the cavity mode for the proper amount of time.

The first step is the preparation of the entangled two-cavity state. This is done by sending a two-level atom, initially in the excited state $|e\rangle$, through the three cavities. The interaction time with the first cavity (C_a) is adjusted so that state $|\Psi_1\rangle = \alpha|e,0,0\rangle - \beta|g,1,0\rangle$ is produced ($|i,n,m\rangle$ denotes atomic state $|i\rangle$ and Fock states $|n,m\rangle$ for cavities C_a and C_b respectively). That is, the atom has a probability $|\alpha|^2$ of remaining in the initial state, and a probability $|\beta|^2$ of decaying to the state $|g\rangle$, leaving one photon in the resonant mode of C_a. The atomic π rotation in C_{aux}, takes $|\Psi_1\rangle$ into $|\Psi_2\rangle = \alpha|g,0,0\rangle + \beta|e,1,0\rangle$, and another π rotation, this time with the resonant mode of C_b, produces the desired entangled state $|\Psi_f\rangle = \alpha|0,0\rangle + \beta|1,1\rangle$ of the two modes. The atom leaves the setup always in the ground state.

Due to cavity losses, the prepared state $|\Psi_f\rangle$ evolves into the mixed state described by Eq. (5). In order to measure the disentanglement of the state of the high-Q cavities, another two-level atom is sent again through the setup. This atom enters C_a in the ground state and undergoes a Rabi π rotation if there is one photon in the resonant mode. Next, C_{aux} is used to generate yet another atomic π rotation and then the atom state undergoes a final rotation in C_b, so that $|g1\rangle \rightarrow (|g1\rangle + |e0\rangle)/\sqrt{2}$, $|e0\rangle \rightarrow (-|g1\rangle + |e0\rangle)/\sqrt{2}$. The rotation in C_{aux} includes a control phase $\delta/2$ ($|g\rangle \rightarrow e^{-i\delta/2}|e\rangle$ and $|e\rangle \rightarrow -e^{i\delta/2}|g\rangle$). Finally, the atomic internal state is measured by ionization. The probability $P_e(\delta,t)$ of finding the atom in the excited state is given by the same expression (9) obtained for the two-ion system, leading to the concurrence in the same way.

Finite-time separability is related, in the example discussed above, to properties of the reservoir and the initial state. The same phenomenon occurs for a diffusive reservoir, acting on the two-qubit system. The corresponding contribution for the master equation can be written in the form (7) with $\hat{c}_1 = \hat{\sigma}_-$ and $\hat{c}_2 = \hat{\sigma}_+$ for each reservoir. Due to the symmetry of this reservoir, all states belonging to the subspace $\{|00\rangle, |11\rangle\}$, or yet $\{|01\rangle, |10\rangle\}$, display finite-time separability, independently of the relative population of the states. This is consistent with the findings in [12], which considered the action of classical noise on a two-qubit system.

Since the work of John Bell [24] the subtle property of entanglement has been subjected to many quantitative tests and has led to intriguing consequences. Understanding the physical meaning of entanglement measures remains however a major challenge. In this paper, we have analyzed an example of a decaying two-qubit system for which it is possible to attribute an operational meaning to an entanglement measure, valid throughout the decay process, and we have proposed an experimental procedure which amounts to a direct measurement of entanglement.

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